Theoretical Analysis of the Falling Cylinder Viscometer for Power Law and Bingham Plastic Fluids

F. J. EICHSTADT and G. W. SWIFT
University of Kansas, Lawrence, Kansas

Theoretical equations and their solutions for the falling cylinder viscometer are presented for the Bingham plastic and power law models.

The purpose of this study was to develop and solve exact falling cylinder viscometric equations for those fluids whose rheological properties are essentially time independent. The falling cylinder viscometer is amenable to theoretical treatment, subject to the usual simplifying assumptions. This viscometer minimizes viscous heating effects, provides a positive means for detecting yield stress, requires a reasonably small amount of sample which can be readily isolated from the surroundings, is easily adapted for operation under extremes of pressure and temperature, and can be fabricated at reasonable cost.

Lohrenz and co-workers (5 to 7) developed the falling cylinder viscometer for Newtonian fluids, and Ashare et al. (*1*) extended the theoretical development of the falling cylinder viscometric equations to the area of non-Newtonian fluids in terms of the power law and Ellis models. Because of certain approximations in the work of Ashare et al., their developments are rigorous only in the limit as the annular gap between the falling cylinder and the tube, within which it is confined, approaches zero. Fredrickson and Bird (*2*) derived and solved the equations for the laminar flow of Bingham plastic and power law fluids in concentric annuli. A combination of their method with that of Lohrenz (*5*) should result in equations for the falling cylinder which are theoretically rigorous for fluids that obey the Bingham plastic or power law models regardless of the size of the annular gap between cylinder and tube.

This paper reports on: (1) The development of the theoretical equations for the falling cylinder viscometer for Bingham plastic and power law fluids which relate the terminal velocity v_t of the falling cylinder to the density difference between the cylinder and the fluid $(\sigma - \rho)$. (2) The solution of the theoretical equations so that, given v_t and $(\sigma - \rho)$, the inverse operation of determining the rheological parameters of the fluid can be accomplished. And (3) the difference between the approximate solution of Ashare et al. and the exact solution of this paper for the power law model as a function of κ, the ratio of cylinder to tube radii, and s, the reciprocal of the power law parameter, n.

The development presented in this paper, like that of Lohrenz (*5*) and of Ashare et al. (*1*) does not treat entrance and exit effects, that is, a falling cylinder of infinite length is assumed. In practice, an approach to this idealized limit can be realized by proper design of the viscometer. It should also be pointed out that the Bingham and power law models are idealized relationships which cannot adequately characterize the flow behavior of real fluids over extended ranges of shear stress. However, the power law model has been demonstrated to be of value in presenting the flow behavior of time independent non-Newtonian fluids over restricted ranges of shear stress.

THEORY

The method used to develop the falling cylinder viscometric equations for power law and Bingham plastic fluids consists of combining a force balance on the body, the equations of motion and continuity for the fluid, and a mass balance relating displaced fluid to that flowing through the annular gap between body and tube, with the pertinent rheological model. Cylindrical coordinates were used, and the usual simplifying assumptions, that is, laminar flow, angular and radial components of the velocity vector equal zero, negligible end effects, zero slip at the walls, steady flow conditions, incompressible fluid, and isothermal flow, were invoked.

Development for Power Law Fluids

Figure 1*a* gives the velocity profile and shear distribution for power law fluids in the falling cylinder viscometer. The expression which relates

$$v_t = f(m, n, \kappa, R, g(\sigma - \rho))$$

where $\sigma - \rho$ is the density difference between viscometer body and test fluid, is derived as follows. The z compo-

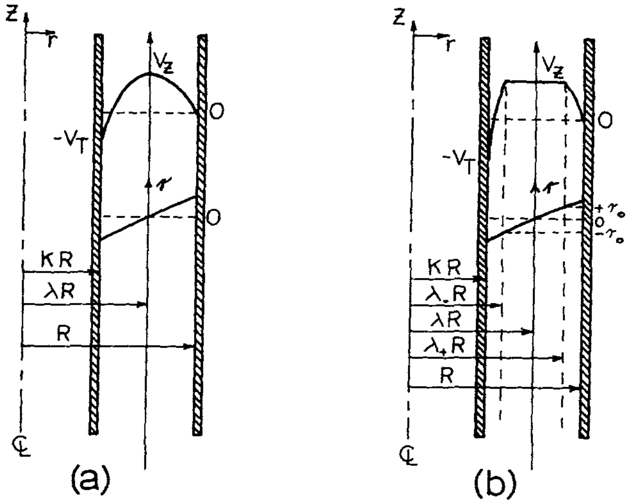

Fig. 1. Velocity profile and shear stress distribution. (a) Power law fluids. (b) Bingham plastic fluids.

nent of the equation of motion becomes

$$d(r\tau_{rz})/dr = \Delta P r/L \quad (1)$$

where

$$\Delta P/L = -dp/dz + \rho g$$

is the net pressure drop per unit length L of the cylinder. Defining $\xi = r/R$ and integrating (1) with the boundary condition

$$\tau_{rz} = 0 \text{ at } r = \lambda R \quad (2)$$

one obtains

$$\tau_{rz} = R\Delta P(\xi - \lambda^2/\xi)/(2L) \quad (3)$$

A force balance over the cylinder gives

(force due to pressure differential) + (buoyancy) + (shear on wall of the cylinder) = (force due to gravity) (4)

or

$$\pi\kappa^2R^2\Delta P + \pi\kappa^2R^2g\rho L + 2\pi\kappa RL(-\tau_w) = \pi\kappa^2R^2g\sigma L \quad (5)$$

Combination of (3) and (5) for $\xi = \kappa$ yields

$$\Delta P/L = (\kappa/\lambda)^2 g(\sigma - \rho) \quad (6)$$

and the insertion of (6) into (3) for $\xi = \kappa$ gives

$$\tau_w = (\kappa/\lambda)^2(\kappa - \lambda^2/\kappa)gR(\sigma - \rho)/2 \quad (7)$$

A mass balance relating the fluid displaced by the movement of the cylinder to that flowing through the annular gap between the cylinder and confining tube gives

$$Q = 2\pi R^2\int_\kappa^1 v_z\xi d\xi = \pi\kappa^2R^2v_t \text{ or } \kappa^2\phi_t/2 = \int_\kappa^1 \phi_z\xi d\xi \quad (8)$$

where ϕ_t and ϕ_z are dimensionless velocities defined by (15). The integration of (8) by parts gives

$$\int_\kappa^1 (d\phi_z/d\xi)\xi^2 d\xi = 0 \quad (9)$$

The power law model for this system may be expressed as

$$\tau_{rz} = -m(dv_z/dr)^n \quad \kappa \leq \xi \leq \lambda \quad (10)$$

$$\tau_{rz} = m(-dv_z/dr)^n \quad \lambda \leq \xi \leq 1 \quad (11)$$

Combining these expressions with (3), one gets

$$dv_z/dr = [R\Delta P/(2mL)]^s(\lambda^2/\xi - \xi)^s \quad \kappa \leq \xi \leq \lambda \quad (12)$$

$$-dv_z/dr = [R\Delta P/(2mL)]^s(\xi - \lambda^2/\xi)^s \quad \lambda \leq \xi \leq 1 \quad (13)$$

where

$$s = 1/n \quad (14)$$

By defining

$$\phi_z = v_z/\{R[R\Delta P/(2mL)]^s\} \quad (15)$$

(12) and (13) may be written as

$$d\phi_z/d\xi = (\lambda^2/\xi - \xi)^s \quad \kappa \leq \xi \leq \lambda \quad (16)$$

$$d\phi_z/d\xi = -(\xi - \lambda^2/\xi)^s \quad \lambda \leq \xi \leq 1 \quad (17)$$

Then (9), (16), and (17) may be combined to give

$$\int_\kappa^\lambda (\lambda^2/\xi - \xi)^s\xi^2 d\xi - \int_\lambda^1 (\xi - \lambda^2/\xi)^s\xi^2 d\xi = 0 \quad (18)$$

Thus, if κ and s are known, one may find

$$\lambda = f(\kappa, s) \quad (19)$$

Integration of (16) and (17) with the boundary conditions

$$\phi_z = -\phi_t \text{ at } \xi = \kappa \quad (20)$$

$$\phi_z = 0 \text{ at } \xi = 1 \quad (21)$$

gives

$$\phi_z + \phi_t = \int_\kappa^\xi (\lambda^2/\xi - \xi)^s d\xi \quad \kappa \leq \xi \leq \lambda \quad (22)$$

$$-\phi_z = \int_\xi^1 -(\xi - \lambda^2/\xi)^s d\xi \quad \lambda \leq \xi \leq 1 \quad (23)$$

But at $\xi = \lambda$, (22) must be equal to (23), so that

$$\phi_t = \int_\kappa^\lambda (\lambda^2/\xi - \xi)^s d\xi - \int_\lambda^1 (\xi - \lambda^2/\xi)^s d\xi \quad (24)$$

giving

$$\phi_t = f(\lambda, \kappa, s) = f(\kappa, s) \quad (25)$$

since $\lambda = f(\kappa, s)$ by (19). Thus, given κ and s, both ϕ_t and λ may be found. Combining (6) and (15) for $\phi_z = -\phi_t$ and $v_z = -v_t$ one obtains

$$v_t = \phi_t R\{(\kappa/\lambda)^2 g(\sigma - \rho)R/(2m)\}^s \quad (26)$$

which is the final working equation relating terminal velocity of the falling cylinder to the density difference by using the power law rheological model.

Given experimental terminal velocity data as a function of the density data for a specific falling body as a plot of ln (v_t) vs. ln $(\sigma - \rho)$, the rheological parameters s and m of the power law model are determined as follows:

1. Determine the slope of the logarithmic plot, thereby establishing the value of s. Use linear regression analysis for this determination.
2. Knowing s and κ, calculate λ from (18) and ϕ_t from (24).
3. Calculate m from (26) by using any pair of v_t and $(\sigma - \rho)$ values on the regression line.

Development for Bingham Plastic Fluids

Figure 1*b* gives the velocity profile and shear distribution for Bingham plastic fluids in the falling cylinder viscometer. λ_- and λ_+ are the limits of plug flow and at $r = \lambda R$, $\tau_{rz} = 0$.

Equations (3) and (6) apply for Bingham plastic fluids but (9) must be written as

$$\int_\kappa^1 (d\phi_{zB}/d\xi)\xi^2 d\xi = 0 \quad (27)$$

where ϕ_{zB} is a dimensionless velocity defined by (30). Also, for Bingham plastic fluids, the following dimensionless variables [analogous to those used by Fredrickson and Bird (2)] are defined:

$$T = 2\tau_{rz}L/(R\Delta P) \quad (28)$$

$$T_o = 2\tau_o L/(R\Delta P) \quad (29)$$

$$\phi_{zB} = 2\mu_o v_z L/(R^2\Delta P) \quad (30)$$

$$\phi_{tB} = 2\mu_o v_t L/(R^2\Delta P) \quad (31)$$

TABLE 1. VALUES OF λ FOR POWER LAW MODEL

s \ κ	0.5	0.6	0.7	0.8	0.9	0.925	0.950	0.975
0	0.8261	0.8476	0.8759	0.9111	0.9526	0.9640	0.9756	0.9877
1	0.7906	0.8246	0.8631	0.9055	0.9513	0.9632	0.9753	0.9876
2	0.7705	0.8124	0.8566	0.9028	0.9507	0.9629	0.9752	0.9875
3	0.7579	0.8049	0.8526	0.9011	0.9503	0.9626	0.9751	0.9875
4	0.7494	0.7998	0.8499	0.9000	0.9500	0.9625	0.9750	0.9875
5	0.7433	0.7962	0.8480	0.8992	0.9498	0.9624	0.9750	0.9875
6	0.7387	0.7935	0.8466	0.8986	0.9497	0.9623	0.9749	0.9875
7	0.7351	0.7913	0.8455	0.8981	0.9496	0.9623	0.9749	0.9875
8	0.7323	0.7896	0.8446	0.8978	0.9495	0.9622	0.9749	0.9875
9	0.7299	0.7883	0.8439	0.8975	0.9494	0.9622	0.9749	0.9875

where μ_o and τ_o are the parameters of the Bingham plastic model.

Consider the plug flow region and make a force balance on it (since v_z is constant, ΔP is also constant). Then

$$\pi(\lambda_+{}^2R^2 - \lambda_-{}^2R^2)\Delta P - 2\pi\lambda_+RL\tau_o - 2\pi\lambda_-RL\tau_o = 0 \quad (32)$$

which, in dimensionless variables, becomes

$$\lambda_- = \lambda_+ - T_o \quad (33)$$

Writing (3) in dimensionless variables and setting $T = T_o$ at $\xi = \lambda_+$, one obtains

$$\lambda^2 = \lambda_+(\lambda_+ - T_o) \quad (34)$$

Combination of (6) and (34) yields

$$\Delta P/L = (\kappa/\lambda_+)^2 g(\sigma - \rho) + 2\tau_o/(R\lambda_+) \quad (35)$$

and at $\xi = \kappa$, (3), (34), and (35) give

$$\tau_w = \{(\kappa/\lambda_+)^2 g(\sigma-\rho) + 2\tau_o/(R\lambda_+)\}(R/2)\{\kappa - (\lambda_+{}^2/\kappa)\} + \lambda_+\tau_o/\kappa \quad (36)$$

The Bingham plastic model for this system may be expressed as

$$\tau_{rz} = -\mu_o dv_z/dr - \tau_o \quad \kappa \leq \xi \leq \lambda_- \quad (37)$$

$$-\tau_o \leq \tau_{rz} \leq +\tau_o \quad \lambda_- \leq \xi \leq \lambda_+ \quad (38)$$

$$\tau_{rz} = -\mu_o dv_z/dr + \tau_o \quad \lambda_+ \leq \xi \leq 1 \quad (39)$$

or, in terms of dimensionless variables, as

$$T = -d\phi_{zB}/d\xi - T_o \quad \kappa \leq \xi \leq \lambda_- \quad (40)$$

$$-T_o \leq T \leq +T_o \quad \lambda_- \leq \xi \leq \lambda_+ \quad (41)$$

$$T = -d\phi_{zB}/d\xi + T_o \quad \lambda_+ \leq \xi \leq 1 \quad (42)$$

Combining (3) and (28) with (40), (41), and (42), one gets

$$d\phi_{zB}/d\xi = -T_o - \xi + \lambda^2/\xi \quad \kappa \leq \xi \leq \lambda_- \quad (43)$$

$$d\phi_{zB}/d\xi = 0 \quad \lambda_- \leq \xi \leq \lambda_+ \quad (44)$$

$$d\phi_{zB}/d\xi = T_o - \xi + \lambda^2/\xi \quad \lambda_+ \leq \xi \leq 1 \quad (45)$$

and if these three equations are combined with (27)

$$\int_\kappa^{\lambda_-} (-T_o - \xi + \lambda^2/\xi)\xi^2 d\xi + \int_{\lambda_+}^1 (T_o - \xi + \lambda^2/\xi)\xi^2 d\xi = 0 \quad (46)$$

Combining (33), (34), and (46), integrating, and simplifying, one obtains

$$0 = -3(1-\kappa^4) + 6\lambda_+(\lambda_+ - T_o)(1-\kappa^2) + 4T_o(1+\kappa^3) - T_o(2\lambda_+ - T_o)^3 \quad (47)$$

Thus, given κ and λ_+, T_o is determined, that is

$$T_o = f(\kappa, \lambda_+) \quad (48)$$

Integration of (43) and (45) with the boundary conditions

$$\phi_{zB} = -\phi_{tB} \text{ at } \xi = \kappa \quad (49)$$

$$\phi_{zB} = 0 \text{ at } \xi = 1 \quad (50)$$

gives

$$\phi_{zB}|_{\lambda_-} + \phi_{tB} = \int_\kappa^{\lambda_-} (-T_o - \xi + \lambda^2/\xi)d\xi \quad (51)$$

$$\phi_{zB}|_{\lambda_+} = \int_{\lambda_+}^1 (T_o - \xi + \lambda^2/\xi)d\xi \quad (52)$$

but $\phi_{zB}|_{\lambda_-} = \phi_{zB}|_{\lambda_+}$ so that (51) and (52) can be equated and integrated to give

$$2\phi_{tB} = 2T_o(1-\lambda_+) - 1 + (T_o+\kappa)^2 + 2\lambda_+(\lambda_+ - T_o)\ln\{(\lambda_+ - T_o)/(\lambda_+\kappa)\} \quad (53)$$

which gives

$$\phi_{tB} = f(T_o, \kappa, \lambda_+) = f(\kappa, \lambda_+) \quad (54)$$

since $T_o = f(\kappa, \lambda_+)$ by (48). From (31) and (35)

$$v_t = \phi_{tB}R^2\{(\kappa/\lambda_+)^2 g(\sigma-\rho) + 2\tau_o/(R\lambda_+)\}/(2\mu_o) \quad (55)$$

which is the final working equation relating terminal velocity of the falling cylinder to the density difference by using the Bingham plastic rheological model. From (29) and (35)

$$\tau_o = \{(\kappa/\lambda_+)^2 T_o Rg(\sigma-\rho)\}/[2(1-T_o/\lambda_+)] \quad (56)$$

When $v_t = 0$, $g(\sigma-\rho) = g(\sigma-\rho)_o$, $\lambda_+ = 1$, $\lambda_- = \kappa$, and, from (33), $T_o = 1 - \kappa$. Then (56) reduces to

$$\tau_o = \kappa(1-\kappa)Rg(\sigma-\rho)_o/2 \quad (57)$$

Thus, τ_o may be found from the intercept of a plot of v_t vs. $(\sigma - \rho)$.

Given experimental terminal velocity data as a function of the density data for a specific falling body as a plot of v_t vs. $(\sigma-\rho)$, the rheological parameters τ_o and μ_o of the Bingham plastic model are determined as follows:

1. Determine the intercept $(\sigma - \rho)_o$ at $v_t = 0$ and calculate τ_o from (57).
2. Select values of T_o and calculate the corresponding λ_+ and ϕ_{tB} values from (47) and (53), respectively.
3. Using the system of values generated in step 2 compute the corresponding $(\sigma - \rho)$ values from (56) and v_t values from (55) with the value of τ_o determined in step 1 and arbitrary values of μ_o.
4. Construct plots of the generated v_t vs. $(\sigma-\rho)$ values which have parameter μ_o. The parametric curve which most nearly coincides with the experimental curve of v_t vs. $(\sigma - \rho)$ has the μ_o value that best describes the fluid.

Development of the Friction Factor

For annular flow of power law fluids, a friction factor–Reynolds number relationship may be developed as follows:

TABLE 2. VALUES OF ϕ_t FOR POWER LAW MODEL

s \ κ	0.5	0.6	0.7	0.8	0.9	0.925	0.950	0.975
0	0.1510	0.9433×10^{-1}	0.5137×10^{-1}	0.2195×10^{-1}	0.5249×10^{-2}	0.2198×10^{-2}	0.1281×10^{-2}	0.3164×10^{-3}
1	0.5822×10^{-1}	0.2736×10^{-1}	0.1072×10^{-1}	0.2978×10^{-2}	0.3513×10^{-3}	0.1462×10^{-3}	0.4274×10^{-4}	0.5274×10^{-5}
2	0.2376×10^{-1}	0.8599×10^{-2}	0.2468×10^{-2}	0.4507×10^{-3}	0.2640×10^{-4}	0.8232×10^{-5}	0.1604×10^{-5}	0.9887×10^{-7}
3	0.1002×10^{-1}	0.2833×10^{-2}	0.6009×10^{-3}	0.7251×10^{-4}	0.2114×10^{-5}	0.4943×10^{-6}	0.6416×10^{-7}	0.1976×10^{-8}
4	0.4330×10^{-2}	0.9631×10^{-3}	0.1517×10^{-3}	0.1213×10^{-4}	0.1763×10^{-6}	0.3091×10^{-7}	0.2674×10^{-8}	0.4122×10^{-10}
5	0.1905×10^{-2}	0.3349×10^{-3}	0.3928×10^{-4}	0.2085×10^{-5}	0.1513×10^{-7}	0.1987×10^{-8}	0.1146×10^{-9}	0.8832×10^{-12}
6	0.8499×10^{-3}	0.1185×10^{-3}	0.1036×10^{-4}	0.3657×10^{-6}	0.1324×10^{-8}	0.1305×10^{-9}	0.5016×10^{-11}	0.1932×10^{-13}
7	0.3836×10^{-3}	0.4249×10^{-4}	0.2775×10^{-5}	0.6512×10^{-7}	0.1177×10^{-9}	0.8700×10^{-11}	0.2228×10^{-12}	0.4285×10^{-15}
8	0.1747×10^{-3}	0.1540×10^{-4}	0.7519×10^{-6}	0.1174×10^{-7}	0.1060×10^{-10}	0.5871×10^{-12}	0.1003×10^{-13}	0.9660×10^{-17}
9	0.8023×10^{-4}	0.5632×10^{-5}	0.2056×10^{-6}	0.1236×10^{-8}	0.9637×10^{-12}	0.4005×10^{-13}	0.4559×10^{-15}	0.2196×10^{-18}

Define

$$f = 2\tau/(\rho\, v_t^2) \qquad (58)$$

and let

$$\tau = R\Delta P(1-\kappa)/(2L) \qquad \text{(see reference 2)} \qquad (59)$$

Then, combination of (6), (58), and (59) gives

$$f = (\kappa/\lambda)^2 R(1-\kappa) g(\sigma-\rho)/(\rho\, v_t^2) \qquad (60a)$$

or, by (15)

$$f = 2m(1-\kappa)\,(v_t/\phi_t R)^n/(\rho v_t^2) \qquad (60b)$$

By convention

$$f = 16/N_{Re} \qquad (61)$$

for laminar flow. Therefore, by combining (60*b*) and (61)

$$N_{Re} = D_e\, \rho\, v_t^{2-n}/m \qquad (62)$$

where

$$D_e = D^n \phi_t^n / 2^{n-3}(1-\kappa) \qquad (63)$$

Note that the N_{Re} defined by (62) has a form similar to the one proposed by Metzner and Reed (8) for non-Newtonian flow in circular conduits.

For Bingham plastic fluids a similar analysis yields

$$N_{Re} = D_{eB}\rho v_t/\mu_o \qquad (64)$$

where

$$D_{eB} = 4D\phi_{tB}/(1-\kappa) \qquad (65)$$

Solutions of Theoretical Equations

The equations to be solved for a power law fluid in the falling cylinder viscometer are (18), (24), and (26). Computations were carried out on the IBM 7040/1401 computer by using two methods as depicted on the flow sheet presented in Figure 2. Method II, which is valid for integer and noninteger values of s, requires that (18) be integrated numerically by using Simpson's rule (4) after making an initial guess for λ. Iteration by the method of *regula falsi* to improve the value of λ is continued until convergence is obtained. Then ϕ_t is determined by numerically integrating (24) by using Simpson's rule. To test the convergence criteria used in method II, an alternate scheme was developed. Method I requires that (18) and (24) be integrated analytically and is, therefore, restricted to integer values of s. In this method the appropriate root of λ was extracted from the integrated form of (18) explicitly for $s = 0$, 1, and 3; however, for $s = 3$ it was more expedient and for $s = 2$ and $s = 4$ necessary to solve (18) by an iterative method such as the Newton-Raphson method for polynomials (3). Because of the difficulties involved in extracting the root of λ for integer values of s greater than 4, it was not possible to test method II against method I for values of s greater than 4. Values computed for λ and ϕ_t by each of the two methods agreed within one in the fifth significant figure for $s = 1$, 2, 3, and 4. The values for λ and ϕ_t, computed by

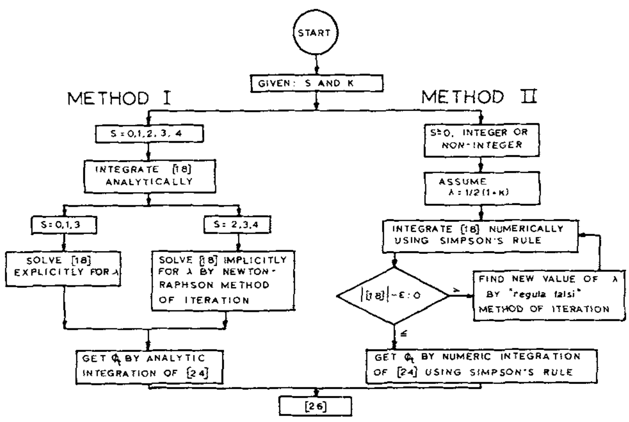

Fig. 2. Flow sheet for solution of power law model equations.

TABLE 3. VALUES OF λ_+ FOR BINGHAM PLASTIC MODEL

T_o \ κ	0.5	0.6	0.7	0.8	0.9	0.925	0.950	0.975
0.01	0.7948	0.8290	0.8677	0.9103	0.9562	0.9681	0.9803	0.9925
0.02	0.7990	0.8334	0.8723	0.9150	0.9611	0.9730	0.9852	0.9975
0.04	0.8075	0.8422	0.8814	0.9244	0.9708	0.9828	0.9951	
0.06	0.8159	0.8510	0.8905	0.9339	0.9805	0.9926		
0.08	0.8244	0.8598	0.8997	0.9433	0.9903			
0.10	0.8328	0.8686	0.9088	0.9528	1.0000			
0.15	0.8539	0.8906	0.9316	0.9764				
0.20	0.8749	0.9125	0.9544	1.0000				
0.30	0.9168	0.9563	1.0000					
0.40	0.9585	1.0000						
0.50	1.0000							

TABLE 4. VALUES OF ϕ_{tB} FOR BINGHAM PLASTIC MODEL

T_o \ κ	0.5	0.6	0.7	0.8	0.9	0.925	0.950	0.975
0.01	0.5656×10^{-1}	0.2636×10^{-1}	0.1019×10^{-1}	0.2756×10^{-2}	0.2988×10^{-3}	0.1171×10^{-3}	0.3010×10^{-4}	0.2279×10^{-5}
0.02	0.5490×10^{-1}	0.2537×10^{-1}	0.9667×10^{-2}	0.2535×10^{-2}	0.2474×10^{-3}	0.8913×10^{-4}	0.1847×10^{-4}	0.2958×10^{-6}
0.04	0.5158×10^{-1}	0.2338×10^{-1}	0.8617×10^{-2}	0.2100×10^{-2}	0.1519×10^{-3}	0.4035×10^{-4}	0.2396×10^{-5}	
0.06	0.4826×10^{-1}	0.2140×10^{-1}	0.7584×10^{-2}	0.1683×10^{-2}	0.7316×10^{-4}	0.8196×10^{-5}		
0.08	0.4495×10^{-1}	0.1944×10^{-1}	0.6576×10^{-2}	0.1291×10^{-2}	0.1971×10^{-4}			
0.10	0.4166×10^{-1}	0.1752×10^{-1}	0.5604×10^{-2}	0.9350×10^{-3}	0.0000			
0.15	0.3364×10^{-1}	0.1292×10^{-1}	0.3392×10^{-2}	0.2578×10^{-3}				
0.20	0.2603×10^{-1}	0.8763×10^{-2}	0.1615×10^{-2}	0.0000				
0.30	0.1278×10^{-1}	0.2444×10^{-2}	0.0000					
0.40	0.3517×10^{-2}	0.0000						
0.50	0.0000							

method II, are reported in Tables 1 and 2, respectively, for $0 \leq s \leq 9$ and $0.5 \leq \kappa \leq 0.975$. Note that λ varies very little with s for κ's approaching 1, which is in agreement with the results of Vaughn (*9*), who studied the possible range of λ with n in concentric fixed wall annuli.

For power law fluids, the approximate equation of Ashare et al. (*1*) was found to agree within 1% with the results of this work for $0 \leq s \leq 3$ and $\kappa \geq 0.9$.

The equations to be solved for a Bingham plastic fluid in the falling cylinder viscometer are (47), (53), (55), and (56). These equations may be solved explicitly for the desired variable except for (47), which must be solved implicitly for λ_+ in the same manner as (18) is solved for λ in method I for power law fluids. Values for λ_+ and ϕ_{tB} as functions of T_o and κ are reported in Tables 3 and 4, respectively.

CONCLUSIONS

1. Exact falling cylinder viscometric equations for Bingham plastic and power law model fluids were developed and solved in this study.
2. The approximate falling cylinder viscometric equation of Ashare et al. (*1*) is less than 1% in error with the rigorous solution for the power law model fluid when considered within the bounds $0 \leq s \leq 3$ and $0.9 \leq \kappa \leq 1$.

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NOTATION

D = diameter of viscometer fall tube, cm.
D_e = equivalent diameter as defined in (63), cm.^{n}
D_{eB} = equivalent diameter as defined in (65), cm.
L = length of viscometer body, cm.
N_{Re} = Reynolds number, dimensionless
Q = volume rate of flow, cc./sec.
R = radius of viscometer fall tube, cm.
T = dimensionless shear stress
T_o = dimensionless yield stress
f = friction factor, dimensionless
g = acceleration due to gravity, cm./sec.^{2}
m = power law model parameter, (dyne) (sec.^{n})/sq. cm.
n = power law model parameter, dimensionless
p = static pressure, dynes/sq. cm.
r = radius, cm.
s = reciprocal of n, dimensionless
t = time, sec.
v_t = terminal velocity, cm./sec.
v_z = velocity in longitudinal direction, cm./sec.

Greek Letters

ΔP = pressure drop, dynes/sq. cm.
ϵ = small positive integer, dimensionless
κ = ratio of body radius to radius of viscometer fall tube, dimensionless
λ = dimensionless radius of maximum velocity
λ_- = dimensionless inner radial limit of plug flow
λ_+ = dimensionless outer radial limit of plug flow
μ = Newtonian viscosity, g./(cm.) (sec.)
μ_o = plastic viscosity, g./(cm.) (sec.)
ξ = dimensionless radius
ρ = fluid density, g./cc.
σ = body density, g./cc.
τ, τ_{rz} = shear stress, dynes/sq. cm.
τ_o = yield stress, dynes/sq. cm.
τ_w = shear stress at inner wall, dynes/sq. cm.
ϕ_t = dimensionless terminal velocity for the power law model
ϕ_{tB} = dimensionless terminal velocity for the Bingham plastic model
ϕ_z = dimensionless velocity in the longitudinal direction for the power law model
ϕ_{zB} = dimensionless velocity in the longitudinal direction for the Bingham plastic model

LITERATURE CITED

1. Ashare, Edward, R. B. Bird, and J. A. Lescarboura, *A.I.Ch.E. J.*, **11**, 910-916 (1965).
2. Fredrickson, A. G., and R. B. Bird, *Ind. Eng. Chem.*, **50**, 349-52 (1958); correspondence, *Ind. Eng. Chem.*, **50**, 1599-1600 (1958).
3. Henrici, P., "Elements of Numerical Analysis," Wiley, New York (1964).
4. Kunz, K. S., "Numerical Analysis," p. 146, McGraw-Hill, New York (1957).
5. Lohrenz, John, Ph.D. thesis, Univ. Kansas, Lawrence (1960).
6. ———, and Fred Kurata, *A.I.Ch.E. J.*, **8**, 190-193 (1962).
7. Lohrenz, John, G. W. Swift, and Fred Kurata, *ibid.*, **6**, 547-550 (1960).
8. Metzner, A. B., and J. C. Reed, *ibid.*, **1**, 434-440 (1955).
9. Vaughn, R. D., *Soc. Petrol. Eng. J.*, 274-276 (Dec., 1963).

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